Inflation, universality and attractors

Scalisi, Marco

Publication date:
2016

3 Inflationary Cosmology

This chapter is devoted to the inflationary paradigm as solution to the standard cosmological puzzles discussed in the previous chapter. We present the basic features of inflation, how this modifies the causal structure of the space-time and its implementation through a scalar field. Then, we discuss the implications of treating this scenario quantum-mechanically: the zero-point fluctuations of the inflaton field become the fundamental origin of those perturbations we can measure in the sky in the form of the CMB temperature anisotropies and primordial gravitational waves. In the end, we present the latest Planck data which provide stringent constraints on the fundamental dynamics of inflation.

3.1 Inflation and the smooth background

The shortcomings of standard cosmology concern the initial conditions of our Universe that require serious fine-tuning in order to reproduce what we observe today. The flatness problem can be solved by assuming that the initial value of the curvature was precisely flat. Similarly, in order to solve the horizon problem, one should imagine at least 10^6 causally disconnected spatial patches to have started their evolution exactly in the same physical conditions, in particular at the same temperature and same magnitude of perturbations. Postulating all this is possible but hardly attractive to a physicist that aims to understand the very early Universe.

In order to do better, inflation was proposed in the 1980's [9–11] to solve these problems all at once. The fundamental idea is that the primordial Universe underwent a finite phase of quasi-exponential expansion (similar to the one we are experiencing nowadays with dark energy) which changed the causal structure and how information propagates. As a bonus, one gets a physical mechanism to explain the presence of very small inhomogeneities as quantum fluctuations in the very early Universe; ultimately, these represent the seeds for the large scale structures we observe in the sky.

3.1.1 Basic idea

Standard cosmology assumes that the early Universe was dominated by some form of energy satisfying the strong energy condition $\rho+3p \geq 0$, which implies a decelerating phase of the scale factor, $\ddot{a} < 0$, as dictated by Eq. (2.17). This is at the core of both the flatness and horizon problems.

Inflation is nothing but inverting such a behavior and postulating a phase of accelerated expansion such as

$$\ddot{a} > 0\,, \tag{3.1}$$

which implies that the Universe was filled with some kind of matter with negative pressure, satisfying

$$\rho + 3p < 0\,. \tag{3.2}$$

The idea that, at very early times, neither matter nor radiation represented the dominant components of energy is not in contrast with any well-tested physical theory. In fact, the standard model of particles physics (SM) cannot be assumed to work up to the first moments after the Big Bang, when energies were several orders of magnitude higher than the domain of validity of the SM (which extends up to around one TeV). Inflation lives off the idea that something non-trivial might have happened due to high-energy physics.

3.1.2 Decreasing Hubble radius

Interestingly, the condition Eq. (3.1) turns out to be equivalent to a decreasing comoving Hubble radius

$$\frac{d}{dt}(aH)^{-1} < 0\,, \tag{3.3}$$

which gives a deeper insight into the causal structure of a Universe undergoing a phase of inflationary expansion. Typical scales, being initially inside the horizon, leaves the radius of causal contact as inflation proceeds and the Hubble radius $(aH)^{-1}$ decreases. They start reentering the horizon when inflation ends, the standard cosmological evolution progresses and $(aH)^{-1}$ increases. This situation is illustrated in Fig. 3.1.

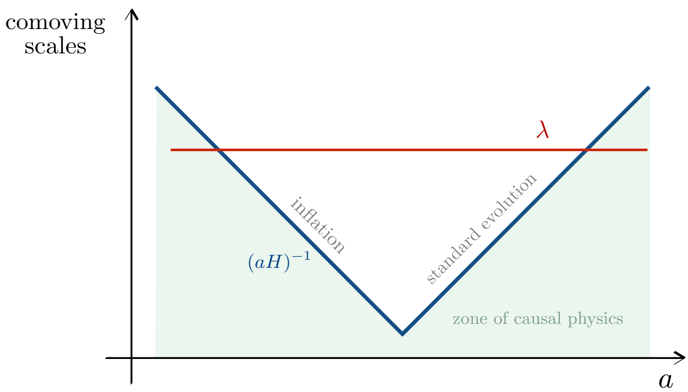

FIGURE 3.1

The comoving Hubbe radius and a typical comoving scale as a function of the scale factor. Due to the anomalous scaling of the comoving Hubble radius, which does not remain constant in time as it happens for all typical scales, the zone of causal physics change with time.

3.1.3 Quasi-de Sitter phase and Hubble flow functions

The rate of change of the Hubble radius with respect to the time t can be expressed also as

$$\frac{d}{dt}(aH)^{-1} = \frac{\epsilon_1 - 1}{a}\,, \tag{3.4}$$

with

$$\epsilon_1 \equiv -\frac{\dot{H}}{H^2} = \frac{d\ln H}{dN}\,, \tag{3.5}$$

and N being the *number of e-folds* defined as

$$dN \equiv d\ln a = -H dt\,, \tag{3.6}$$

where we have assumed that N decreases as time progresses (one can find also the opposite convention in literature). Then, inflation takes place as long as

$$\epsilon_1 < 1\,, \tag{3.7}$$

which means that the Hubble parameter must vary very slowly in time. The inflationary phase corresponds to a quasi-exponential expansion with H almost constant for the whole duration of the process. The extreme limit $\epsilon_1 = 0$ is an exact de Sitter phase sourced by an pure cosmological constant and then defined by Eq. (2.22) for $w = -1$. However, we know inflation must end in order to give rise to the standard cosmological evolution. This point is defined by $\epsilon_1 = 1$.

In order to assure inflation to last long enough, thus solving the standard shortcomings, we define a second important parameter controlling the duration of the process (in the next subsection, we quantify the amount of inflation needed in order to solve the cosmological puzzles). This is[1]

$$\epsilon_2 \equiv \frac{d\ln\epsilon_1}{dN}\,. \tag{3.8}$$

The condition $|\epsilon_2| < 1$ basically means having a small fractional variation of ϵ_1 which guarantees that inflation persists enough time.

We can further proceed constructing the entire tower of the so-called *Hubble flow functions* ϵ_i defined iteratively as [36, 37]

$$\epsilon_{i+1} \equiv \frac{d\ln|\epsilon_i|}{dN}\,, \tag{3.9}$$

with the first of these quantities identical to the Hubble parameter, $\epsilon_0 = H$.

3.1.4 Puzzles resolution and the amount of inflation

The horizon problem is solved if one allows for enough inflation such that also the largest scales we observe in the sky today (CMB and LSS scales) were inside the horizon at early times. Quantitatively, this means that the

[1] Unfortunately, in the scientific literature, there is no unique convention regarding the symbols identifying the parameters which control the dynamics of inflation. Different authors may assign different symbols to the same parameter. Specifically, ϵ_1 and ϵ_2 are often referred to as ϵ and η in other references (e.g. in [25]). However, throughout the thesis, we will reserve the symbols ϵ and η for the slow-roll parameters later introduced in Sec. 3.1.5.

comoving scales of the observable Universe today $(a_0 H_0)^{-1}$ must fit inside the comoving Hubble radius at the beginning of inflation $(a_i H_i)^{-1}$, that is

$$(a_i H_i)^{-1} > (a_0 H_0)^{-1} \,. \tag{3.10}$$

The amount of inflation needed to allow for this resolution is conveniently quantified by the number of e-folds N defined by Eq. (3.6) and determined by the increase of the scale factor during inflation. Specifically, a total number $N \gtrsim 50-60$ suffices to explain the thermalization of the largest observational scales at present.

A rough estimate can be obtained by assuming that the Universe has been dominated mainly by radiation since the end of inflation (at that moment, the comoving Hubble radius was equal to $(a_e H_e)^{-1}$). This implies that the Hubble parameter scales as $H \propto a^{-2}$. Then we have

$$\frac{a_0 H_0}{a_e H_e} \sim \frac{a_e}{a_0} \sim \frac{T_0}{T_e} \sim 10^{-28} \,, \tag{3.11}$$

where we have assumed $T_0 = 10^{-3}$ eV, as the CMB temperature measured today, and $T_e = 10^{15}$ GeV as the typical expected inflationary energy. Then, Eq. (3.10) becomes

$$(a_i H_i)^{-1} > 10^{28} (a_e H_e)^{-1} \,, \tag{3.12}$$

which means that the Hubble radius had to shrink 28 orders magnitude in order to solve the horizon problem. Since during inflation H is almost constant, we have $H_i \approx H_e$ and then

$$\frac{a_e}{a_i} > 10^{28} \,, \tag{3.13}$$

which, using Eq. (3.6), implies

$$N > 64 \,. \tag{3.14}$$

The flatness problem is overcome by means of the same mechanism. A decreasing comoving Hubble radius $(aH)^{-1}$ drives the value of the total energy density Ω to unity, providing a physical explanation for this apparently fine-tuned configuration. After inflation, the curvature will start diverging from $\Omega \approx 1$, as it happens in a Universe filled with ordinary matter. Interestingly, the same amount of inflation needed to solve the horizon problem is enough to explain the flatness we observe today. In fact, during inflation we have

$$\Omega - 1 = \frac{\kappa^2}{(aH)^2} \propto e^{-2N} \to 0 \,. \tag{3.15}$$

The same number of e-folds quoted before would give the accuracy required for the value observed today.

3.1.5 Scalar field dynamics and slow-roll inflation

The Einstein equations tell us that inflation should be supported by some form of matter with a negative pressure, as given by Eq. (3.2). However, we are still left with the issue of identifying the origin of such an incredible energy which led the scale factor to increase by an order of 10^{28}.

The simplest example is to imagine that (a small portion of) the primordial Universe is filled with a scalar field, often called *inflaton* field, minimally coupled to gravity with Lagrangian

$$\mathcal{L} = \sqrt{-g}\left[\tfrac{1}{2}R - \tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right], \tag{3.16}$$

leading to the energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left[\tfrac{1}{2}\partial^\sigma\phi\,\partial_\sigma\phi + V(\phi)\right]. \tag{3.17}$$

In the case of a homogeneous scalar field $\phi(t)$ filling a patch of the Universe with flat FLRW metric (2.10), the energy density and pressure turn out to be simply

$$\rho \equiv T_{00} = \tfrac{1}{2}\dot{\phi}^2 + V(\phi)\,, \quad p \equiv T_{ii} = \tfrac{1}{2}\dot{\phi}^2 - V(\phi)\,. \tag{3.18}$$

The dynamics and interaction of the spacetime metric and scalar field is described by the two equations

$$H^2 = \frac{1}{3}\left[\frac{\dot{\phi}^2}{2} + V(\phi)\right], \quad \ddot{\phi} + 3H\dot{\phi} + V' = 0\,, \tag{3.19}$$

where primes denote derivatives with respect to ϕ. The first is simply the Friedmann equation (2.17), with $\kappa = 0$. The second is the equation of motion for the scalar field which is derived by varying its action. It describes a particle rolling down along its potential and subject to a friction due to the expansion term $3H\dot{\phi}$. The second Friedmann equation (2.18) simply becomes

$$\dot{H} = -\frac{\dot{\phi}^2}{2}\,, \tag{3.20}$$

which implies that the time evolution of the Hubble parameter depends on the kinetic energy of the field. Alternatively, it is possible to obtain the second equation of (3.19) by taking the time derivative of the first equation and combining this with Eq. (3.20).

This region of the Universe inflates if the state parameter $w = p/\rho < -1/3$, which is easily realizable if the potential energy dominates over the kinetic energy, that is

$$V(\phi) \gg \dot{\phi}^2\,. \tag{3.21}$$

The regime described by Eq. (3.21) is said *slow-roll inflation* as the field will evolve really slowly with respect to the quasi-exponential growth of the scale factor. Further, in order to have an inflationary period lasting long enough, one must ensure a small acceleration of the field and therefore impose

$$|\ddot{\phi}| \ll |3H\dot{\phi}|\,. \tag{3.22}$$

Intuitively, such a scenario is possible any time that the shape of the potential is sufficiently flat (in some measure) as it is shown in the cartoon of Fig. 3.2.

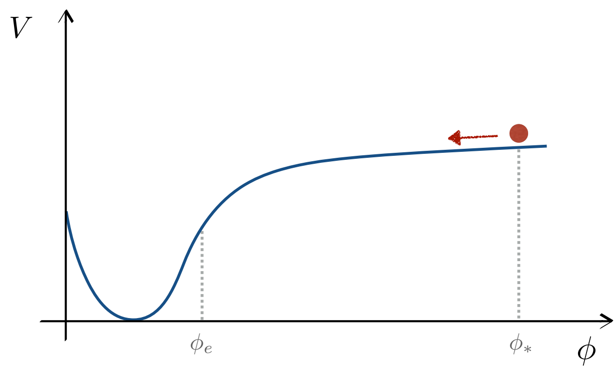

FIGURE 3.2
Cartoon picture of a typical inflationary potential. The scalar field slowly rolls down along the shape driving the quasi-exponential expansion. Inflation ends at ϕ_e and starts at ϕ_*, at least around 60 e-foldings before the end.

Within the slow-roll regime, the dynamical equations (3.19) become

$$H^2 \approx \frac{V(\phi)}{3} \approx \text{constant}\,, \qquad \dot{\phi} \approx -\frac{V'}{3H}\,. \tag{3.23}$$

Given a scalar field with its potential $V(\phi)$, one can verify whether such scenario is suitable for inflation or not by calculating the so-called *slow-roll parameters*, defined as

$$\epsilon \equiv \frac{1}{2}\left(\frac{V'}{V}\right)^2\,, \qquad \eta \equiv \frac{V''}{V}\,, \tag{3.24}$$

and check that

$$\{\epsilon, |\eta|\} \ll 1\,, \tag{3.25}$$

which is equivalent to Eq. (3.21) and Eq. (3.22).

Strictly within the slow-roll approximation, the slow-roll parameters are related to the Hubble flow functions through the following

$$\epsilon_0 = (V/3)^{1/2}\,, \quad \epsilon_1 = \epsilon\,, \quad \epsilon_2 = -4\epsilon + 2\eta\,. \tag{3.26}$$

Eventually, inflation must end and give way to the standard cosmological evolution (with an increasing Hubble radius and ordinary matter domination). This happens when the conditions (3.25) are violated: the trajectory becomes first too steep and the inflaton eventually falls into a local minimum. The oscillations around the vacuum convert the inflationary energy into ordinary particles, within a process called *reheating* (see [38] for a review on this topic and [39] for a recent work).

3.2 Inflation and the background perturbations

3.2.1 The inhomogeneous Universe

The inflationary paradigm elegantly solves the standard cosmological puzzles, providing a natural explanation for the homogeneity and isotropy at large distances. However, at scales smaller than 100 Mpc, we do observe structures in form of galaxies, stars and so on. The standard cosmological theory allows us to accurately trace the evolution of such structures back in time. We are able to identify their origin in the gravitational instability of small density perturbations of a primordial plasma made up of photons and baryons, which have evolved into the large-scale structures of the present Universe.

This idea of structure formation is confirmed by the oldest snapshot we have of our Universe: the cosmic microwave background (CMB). It was produced at the time when electrons and nuclei have just recombined, around 300.000 years after the Big Bang, leaving the CMB photons to freely stream. The tiny temperature fluctuations of order $\delta T/T \sim 10^{-5}$, indicated in Fig. 3.3, reflect the presence of regions with slightly different densities; the wavelength of the photons is red-shifted or blue-shifted depending on the value of the local density. Indeed the properties of the CMB can be time-evolved into a forecast for the Universe that has an excellent match with our observed one.

Despite the stunning success of the theory of structure formation, we are left with some puzzling questions: *what set those initial density perturbations? Which is their fundamental origin? Why were they there at all?*

Surprisingly, inflation suggests a possible answer that is in excellent agreement with observations, thus definitively establishing itself as the leading paradigm for the understanding of the early Universe physics. This answer

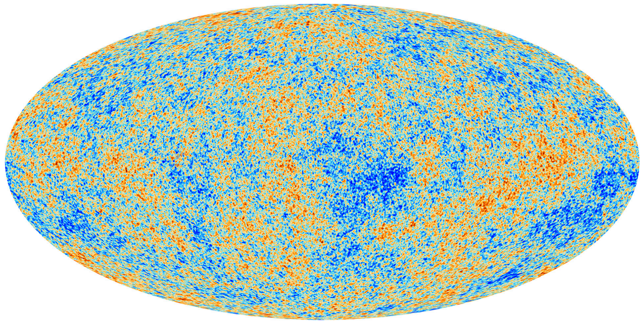

FIGURE 3.3
The fluctuations of 1 part in 10^5 *around the average temperature of* $T = 2.73$ *of the CMB.*

stems from adding quantum mechanics to the fundamental inflationary dynamics. The scalar field implementation provides once more a very useful stage in order to discuss such a physics. In fact, quantum fluctuations $\delta\phi$ are unavoidable in the homogeneous background represented by $\phi(t)$. These source metric perturbations via the Einstein equations and vice versa according to the following scheme

$$\phi(t, \mathbf{x}) = \phi(t) + \delta\phi(t, \mathbf{x}) \quad \Leftrightarrow \quad g_{\mu\nu}(t, \mathbf{x}) = g_{\mu\nu}(t) + \delta g_{\mu\nu}(t, \mathbf{x})\,, \tag{3.27}$$

where $g_{\mu\nu}(t)$ is simply the unperturbed FLRW metric, as given by Eq. (2.10). Due to the symmetries and gauge invariance of the coupled system, the resulting physical perturbations reduce to a scalar and a tensor one (vector perturbations decay during the quasi-exponential expansion). Intuitively, quantum fluctuations excite all the light particles, in the minimal scenario being the inflaton and the graviton. The scalar perturbations couple to the energy density and eventually lead to the inhomogeneities and anisotropies observed in the CMB. The tensor perturbations are often referred to as primordial gravitational waves. They do not couple to the density but induce polarization in the CMB spectrum [40–45]. This is considered to be a unique signature of inflation and many current experiments are searching for it in the sky.

A detailed treatment of the cosmological perturbations theory goes beyond the aim of the present thesis. The interested reader might consult the references [22, 23, 46–48]. In the following, we would like just to sketch the main consequences of a consistent quantum formulation of the inflationary paradigm. In order to simplify the discussion, we will firstly discuss the pure de Sitter and massless case. In the next Sec. 3.3, we will focus on the proper inflationary analysis, regarded as a small deviation from the case studied here, and eventually extrapolate the significant observational parameters.

3.2.2 Quantum scalar fluctuations during inflation

Scalar fluctuations can be fully attributed to the quantum nature of the inflaton field living in an unperturbed FLRW background. This corresponds to a specific gauge (usually called *spatially flat slicing*) where metric perturbations are set equal to zero. It is a perfectly consistent choice in order to discuss the relevant physics and show how scalar fluctuations behave in an inflationary background metric. The decreasing Hubble radius $(aH)^{-1}$ will play again a crucial role, as we will see.

Let us consider the inflaton field $\phi(t, \mathbf{x})$ with a small spatial dependence as given by Eq. (3.27). The corresponding equation of motion is

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2}{a^2}\phi + V' = 0\,, \tag{3.28}$$

which differs from the homogeneous equation (3.19) of the background field $\phi(t)$ for the third extra term. We can Fourier-expand the fluctuations such as

$$\delta\phi(t, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \delta\phi_k(t) e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{3.29}$$

with $\mathbf{x}$ and $\mathbf{k}$ being respectively the comoving coordinates and momenta. Note that the Fourier modes $\delta\phi_k$ depend just on the modulo $k = |\mathbf{k}|$ because of the isotropy of the background metric. Then, we can perturb at first order Eq. (3.28), plug the decomposition (3.29) in and get

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \frac{k^2}{a^2}\delta\phi_k = 0\,, \tag{3.30}$$

where we have neglected the additional term $V''\delta\phi_k$ due to the slow-roll conditions Eq. (3.25) during inflation. Eq. (3.30) can be rewritten in a simpler form, without the Hubble friction term, once we introduce the variable

$$v_k \equiv a\delta\phi_k\,, \tag{3.31}$$

and switch to the conformal time τ. This was defined by Eq. (2.11) and it is naturally related to the comoving Hubble radius as

$$\tau = -\frac{1}{aH}\,, \tag{3.32}$$

during a perfect exponential expansion with H constant. Then, the dynamics of the scalar perturbations can be described simply by the equation of a collection of independent harmonic oscillators

$$\frac{d^2}{d\tau^2} v_k + \omega_k^2(\tau) v_k = 0 \,, \tag{3.33}$$

with time-dependent frequencies

$$\omega_k^2(\tau) = k^2 - \frac{2}{\tau^2} = k^2 - 2(aH)^2 \,. \tag{3.34}$$

The quantization of the physical system now becomes very easy and one proceeds as in the case of the simple harmonic oscillator, following the canonical procedure. In particular, the modes v_k become nothing but the coefficients of the decomposition of the quantum operator

$$\hat{v}(\tau, \mathbf{k}) = v_k(\tau) \hat{a}_{\mathbf{k}} + v_k^*(\tau) \hat{a}_{\mathbf{k}}^\dagger \,, \tag{3.35}$$

where the creation and annihilation operators satisfy the canonical commutation relation

$$\left[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger \right] = \delta^3 \left(\mathbf{k} - \mathbf{k}' \right) \,. \tag{3.36}$$

The quantum zero-point fluctuations are given by

$$\left\langle 0 \,\middle|\, \hat{v}^\dagger(\tau, \mathbf{k}) \hat{v}(\tau, \mathbf{k}') \,\middle|\, 0 \right\rangle = |v_k(\tau)|^2 \delta^3 \left(\mathbf{k} - \mathbf{k}' \right) \tag{3.37}$$

where the vacuum is defined by $\hat{a}_{\mathbf{k}} |0\rangle = 0$ for any $\mathbf{k}$. Therefore, computing the quantum perturbations of the inflaton field reduces to solving the classical equation (3.33) and, then, extracting the time dependence of the Fourier modes $v_k(\tau)$.

The physics of the mode functions v_k, during inflation, is non-trivial and crucially depends on the fact that the comoving Hubble radius shrinks with time. In fact, fluctuations are produced on every scale λ and therefore with any momentum k. While initially being inside the horizon, they leave the zone of causal physics at one point of the accelerated expansion, as schematically shown in Fig. 3.1.

One can prove that an exact solution of Eq. (3.33) is

$$v_k(\tau) = \alpha \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau} \right) + \beta \frac{e^{ik\tau}}{\sqrt{2k}} \left(1 + \frac{i}{k\tau} \right) \,, \tag{3.38}$$

where α and β are some free parameters to be set by means of the initial conditions. These are defined at very early times, when the relevant scales were still inside the horizon. In the *sub-horizon limit* ($k \ll aH$), that is when $k|\tau| \to \infty$, the frequencies (3.34) become time-independent and Eq. (3.33) reduces to

$$\frac{d^2}{d\tau^2}v_k + k^2 v_k = 0\,, \tag{3.39}$$

basically the one of a simple harmonic oscillator. We can exploit this fact in order to get the correct normalized solution

$$\lim_{k|\tau|\to\infty} v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}\,, \tag{3.40}$$

which comes from the requirement of a unique vacuum (so-called *Bunch-Davies* vacuum) being the ground state of energy. This sets $\alpha = 1$ and $\beta = 0$ in Eq. (3.38), thus yielding the definitive expression for the Fourier modes

$$v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right)\,. \tag{3.41}$$

Once we have the complete solution Eq. (3.41), we are particularly interested in studying when the modes leave the horizon. We would like indeed to understand how they behave after inflation and affect late time physics. How can quantum fluctuations, produced during inflation, source density perturbation at CMB decoupling? These events are separated by a huge amount of time where physics is very uncertain. Fortunately, something special happens as we explain below.

The *super-horizon limit* ($k \gg aH$), that is when $k|\tau| \to 0$, corresponds to the solution

$$\lim_{k|\tau|\to 0} v_k = -\frac{i}{\sqrt{2}k^{3/2}\tau}\,. \tag{3.42}$$

Since the conformal time is related to the scale factor by Eq.(2.11), the latter represents a growing mode $v_k \propto\ a$, in de Sitter background. Switching to the physical scalar perturbations by means of Eq. (3.31), one obtains that the amplitude $\delta\phi_k$ remains constant as long as the Hubble radius is smaller than their typical length. Modes freeze outside the horizon and this is a crucial result in order to connect the physics of the early Universe to the time when the density perturbations are created. It is a great bonus we get from inflation as we do not need to worry about the time evolution of such fluctuations for a very substantial part of the cosmic evolution.

Now we can return to Eq. (3.37) and properly evaluate the dimensionless *power spectrum* Δ_v^2 of the quantum fluctuations v_k, defined as

$$\left\langle 0 \left| \hat{v}^\dagger(\tau,\mathbf{k})\hat{v}(\tau,\mathbf{k}') \right| 0 \right\rangle \equiv \frac{2\pi^2}{k^3}\Delta_v^2(k)\ \delta^3\left(\mathbf{k}-\mathbf{k}'\right)\,. \tag{3.43}$$

Then, the power spectrum of the fluctuations after horizon crossing is

$$\lim_{k|\tau|\to 0} \Delta_v^2(k) = \frac{k^3}{2\pi^2}|v_k|^2 = \left(\frac{aH}{2\pi}\right)^2 , \tag{3.44}$$

where we have used Eq. (3.37) in the first step while Eq. (3.42) and Eq. (3.32) in the last. Therefore, the power spectrum of the physical fluctuations of the inflaton field on super-horizon scales is

$$\Delta_{\delta\phi}^2(k) = \left(\frac{H}{2\pi}\right)^2 , \tag{3.45}$$

which is scale-invariant as no k-dependence enters the expression above. Note that this result was first derived in [49], in a perfect de Sitter approximation, before inflation was proposed. A proper inflationary analysis would bring corrections of order $\mathcal{O}(\epsilon, \eta)$. This is shown below in Sec. 3.3.

3.2.3 Classical curvature and density perturbations

In the previous section, we have learned that quantum fluctuations, produced during inflation, stop oscillating once they are stretched to super-horizon scales. Their amplitude freezes at some nonzero value, with scale invariant power spectrum given by Eq. (3.45). This situation lasts for a very long period until the point when the modes re-enter the horizon, during the standard cosmological evolution, as schematically shown in Fig. 3.1. At horizon re-entry, the amplitude of the modes starts oscillating again inducing the density perturbations. However, the energy density directly interacts with the gravitational potential. Therefore, *how do quantum fluctuations of the inflaton affect the metric curvature and ultimately become density perturbations?* Here, we present a very simple and heuristic derivation, mainly based on the *time-delay formalism* developed in [50].

The presence of quantum fluctuations $\delta\phi(t, \mathbf{x})$ over the smooth background $\phi(t)$ translates into local differences δN of the duration of the inflationary expansion, directly related to curvature perturbations ζ. In fact, not every point in space will end inflation at the same time thus leading to local variations of the scale factor a. Then, fluctuations $\delta\phi$ induce curvature perturbations equal to

$$\zeta = \delta N = H\frac{\delta\phi}{\dot{\phi}} = \frac{\delta a}{a} \,. \tag{3.46}$$

The corresponding dimensionless power spectrum is

$$\Delta_\zeta^2(k) = \frac{H^2}{\dot{\phi}^2}\Delta_{\delta\phi}^2(k) = \frac{H^4}{4\pi^2\dot{\phi}^2} \,, \tag{3.47}$$

which, during slow-roll, reads

$$\Delta_\zeta^2 = \frac{1}{12\pi^2}\frac{V^3}{V'^2} = \frac{1}{24\pi^2}\frac{V}{\epsilon}\,, \tag{3.48}$$

where we have used Eq. (3.23) in the first equality and Eq. (3.24) in the second one.

Once inflation ends and the standard cosmological history begins, the energy density will evolve as $\rho = 3H^2$ and, then, decrease as given by Eq. (2.21) (the evolution is shown in Fig. 2.8). Local delays of the expansion lead to local differences in the density, schematically being $\delta N \sim \delta\rho/\rho$. The amplitude of the density fluctuations will be directly related to the amplitude of the curvature perturbations with power spectrum Eq. (3.48).

3.2.4 Primordial gravitational waves

Primordial quantum fluctuations excite also the graviton, corresponding to tensor perturbations δh of the metric. These have two independent and gauge-invariant degrees of freedom, associated to the polarization components of gravitational waves (usually denoted by h_+ and $h_\times$). One can prove that the Fourier modes of these functions satisfy an equation analogous to Eq. (3.30). Therefore, one may proceed identically to what done in Sec. 3.2.2. The dimensionless power spectrum turns out to be

$$\Delta_h^2(k) = 2 \times 4 \times \left(\frac{H}{2\pi}\right)^2\,, \tag{3.49}$$

where the factor 2 is due to the two polarizations and the factor 4 is related to different normalization.

3.3 Inflation and observations

The last 50 years have seen extraordinary success in the development of observational techniques and in the experimental confirmation of our cosmological theories. The discovery of the CMB in 1965 [51] gave the start to a new scientific era where our most speculative ideas have found empirical verification. Analyzing this primordial light has become our fundamental tool for the investigation of the very early Universe physics.

The CMB is essentially the farthest point we can push our observations to. It is nothing but an almost isotropic 2D surface surrounding us and beyond which nothing can directly reach our telescopes. One can draw an analogy to the surface of the Sun: the inner dense plasma does not allow any light to freely stream outwards and the analysis of the last scattering photons (around 8 minutes old) becomes essential in order to probe the internal structure. In fact, the homogeneity and isotropy of the CMB together with its tiny and characteristic temperature anisotropy (see Fig. 3.3) naturally led us to consider inflation as what lies beyond that last scattering surface, around 13.8 billions years old.

Via CMB measurements, we are able to probe the inflationary era and set stringent constraints on the fundamental dynamical mechanism. In the language of the scalar field implementation, we can use observational inputs to impose restrictions on the form of the scalar potential $V(\phi)$. The reason why we are able to have access to such a primordial era is closely connected to the mechanism outlined in the previous section: fluctuations produced during inflation freeze outside the horizon thus providing a link between two very separated moments in time. This situation is depicted in Fig. 3.4.

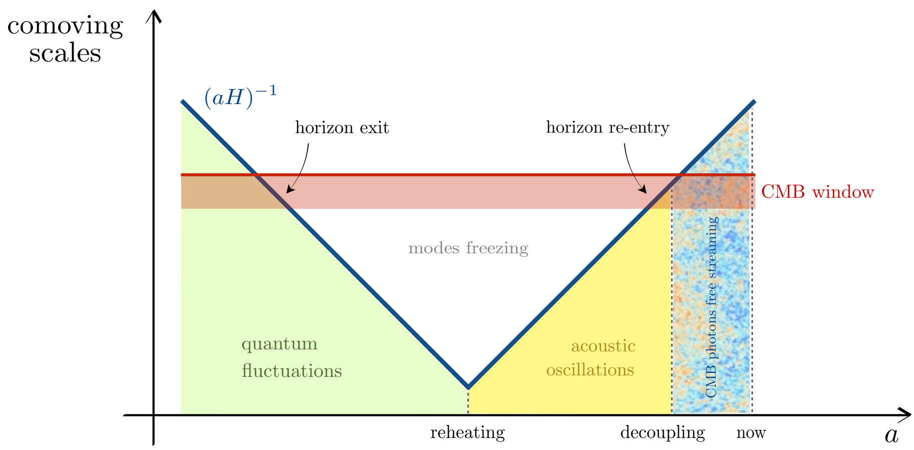

Figure 3.4

Quantum fluctuations produced during inflation (green area) freeze at the horizon exit. They reenter the horizon after reheating thus sourcing acoustic oscillations of the plasma (yellow part). At decoupling time, the CMB photons freely stream towards us who measure their power spectrum just in the small red window.

3.3.1 CMB power spectrum and inflationary observables

The power spectrum of the temperature fluctuations in the CMB contains valuable information on the dynamics of inflation. The characteristic shape is simply dictated by the two-point correlation function of the inflaton fluctuations calculated in Sec. 3.2. A proper investigation of the CMB physics is required in order to understand the functional form. However, this goes beyond the scope of the present work (see e.g. [22, 52] for a detailed treatment). In practice, it is the so-called *transfer function* which relates the two power spectra: it contains all the information regarding the evolution of the initial fluctuations from the moment when they re-enter the horizon to the time of photon-decoupling (yellow part in Fig. 3.4) and, subsequently, their projection in the sky as we observe them today. The final result is the solid line of Fig. 3.5 with the peculiar Doppler peaks originated from the acoustic oscillations of the baryon-photon plasma. The first peak corresponds to a mode that had just time to compress once before decoupling. The other peaks underwent more oscillations and, on small scales, are damped. The high suppression of the power spectrum, at small angular scales, reflects why we are able to probe just a small window of the inflationary era. In terms of the number of e-folds this corresponds to about $\Delta N \approx 7$. On the contrary, scales to the left of the first peak show no oscillations as they were superhorizon at the time of decoupling, and hence have not experienced any oscillations.

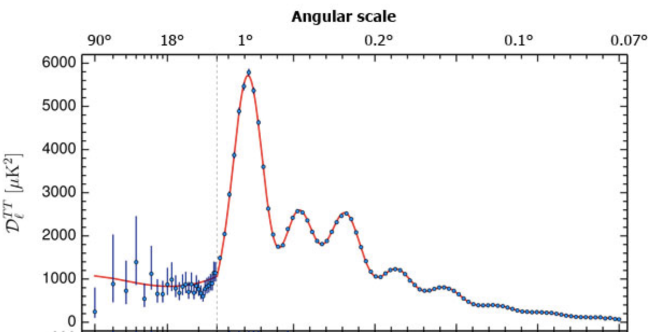

FIGURE 3.5
Power spectrum of the CMB temperature anisotropy as measured by Planck 2015 [13, 14].

In Sec. 3.2, we have derived the power spectrum of perturbations in a perfect de Sitter ($H \approx$ const) and massless ($V'' \approx 0$) approximation. However, an appropriate inflationary analysis would bring some corrections (order slow-roll) and hence a small k-dependence. This is because, during inflation, the energy scale (set by H) will slightly change together with time and the inflaton mass is non-zero, although being very small (order η). In order to parametrize the deviation from scale-invariance, we introduce the *spectral* *indexes* n_s and n_t defined by

$$n_s - 1 \equiv \frac{d \ln \Delta^2_\zeta}{d \ln k}\,, \qquad n_t \equiv \frac{d \ln \Delta^2_h}{d \ln k}\,, \tag{3.50}$$

respectively for scalar and tensor perturbations. In terms of the slow-roll parameters, they read

$$n_s - 1 = 2\eta - 6\epsilon\,, \qquad n_t = -2\epsilon\,. \tag{3.51}$$

Furthermore, since observations probe just a limited range of k, we can express the deviation from scale-invariance by means of the power laws

$$\Delta^2_\xi(k) = \Delta^2_\zeta(k_0)\left(\frac{k}{k_0}\right)^{n_s-1}\,, \qquad \Delta^2_h(k) = \Delta^2_h(k_0)\left(\frac{k}{k_0}\right)^{n_t}\,, \tag{3.52}$$

where k_0 is a normalization point called *pivot scale*. Note that we have only included the first coefficients of scale-dependence; higher-order effects lead to a scale dependence of these coefficients themselves (referred to as running). Finally, the *tensor-to-scalar ratio* is defined by

$$r \equiv \frac{\Delta^2_h(k_0)}{\Delta^2_\zeta(k_0)} = 16\epsilon\,, \tag{3.53}$$

and indicates the suppression of the power of tensor with respect to scalar modes.

3.3.2 Planck data

The Planck satellite [13, 14] has mapped the Universe with unprecedented accuracy. In this way, it has set stringent constraints on the parameters related to the inflationary dynamics. First of all, at $k_0 = 0.05$ Mpc^{-1}, the experimental value for the scalar amplitude (first detected by COBE [53]) is

$$\Delta^2_\zeta(k_0) = (2.14 \pm 0.10) \times 10^{-9}\,. \tag{3.54}$$

Secondly, the deviation from perfect scale-invariance has been definitively confirmed and the scalar spectral index n_s has been measured to be [13,14,54]

$$n_s = 0.968 \pm 0.006 \;(68\%\mathrm{CL})\,. \tag{3.55}$$

On the other hand, the value of the tensor-to-scalar ratio has been observationally bounded to be [54, 55]

$$r < 0.07 \quad (95\%\mathrm{CL})\,. \tag{3.56}$$

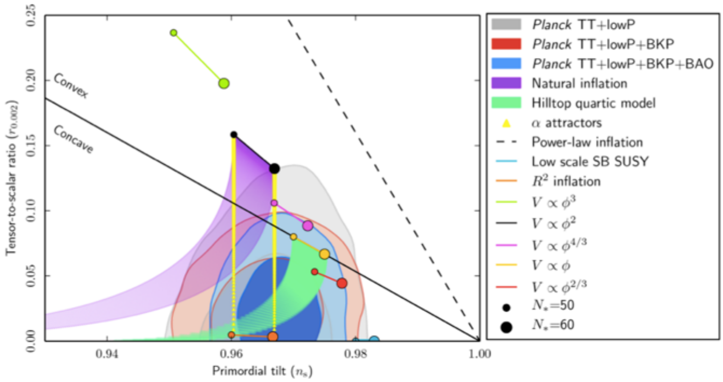

FIGURE 3.6
Planck 2015 results [13, 14] for the spectral index and tensor-to-scalar ratio with the predictions of different inflationary models superimposed.

These can be read in Fig. 3.6, where the predictions of different models of inflation are superimposed[2] and in Fig. 3.7, where the constraints have been improved once including the 95 GHz data from Keck array [55].

Finally, one can get an upper bound on the value of the inflationary energy at horizon crossing. This is given by

$$V^{1/4} \simeq 1.93 \times 10^{16} \left(\frac{r}{0.12}\right)^{1/4} \text{GeV}\,, \tag{3.57}$$

which is obtained by combining Eq. (3.48) and Eq. (3.54). This value indicates that inflation should happen at very high energies (i.e. around $10^{15} - 10^{16}$ GeV), below the Planck scale though. The quartic root dependence on r implies that even a very small value of the tensor-to-scalar ratio (e.g. 10^{-3} or 10^{-5}) does not correspond to a relevant decrease of energy.

[2]Given a potential $V(\phi)$, one can calculate the observational predictions n_s and r by means of the formulas (3.51), (3.53) and (3.24).

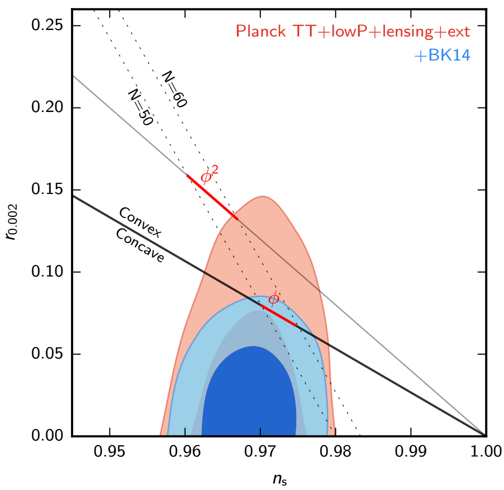

FIGURE 3.7

Improved observational constraints for the spectral index and tensor-to-scalar ratio, after including the 95 GHz data from Keck array [55]. The quadratic model of inflation is basically ruled out.